Fokker Planck and Master equation

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Connections

- Temporal evolution of the probability
- Hypothesis for the Chapman-Kolmogorov equation
- Meaning of the master equation
- The master equation is equivalent to a stochastic differential equation
- Derivation of the Fokker-Plank from a Langevin equation
Transition Probability

Transition probability at a time $\tau$ between two states $y_3 \rightarrow y_2$ in a $\tau$ expansion

$$P_\tau(y_3|y_2) = \delta(y_3 - y_2) + W(y_3|y_2)\tau + O(\tau^2)$$

$W(y_3|y_2) = d_tP(y_3|y_2)$ transition rate

Normalizing: $\alpha(y_2) = \int W(y_3|y_2)dy_3$

$$P_\tau(y_3|y_2) = (1 - \tau\alpha(y_2))\delta(y_3 - y_2) + W(y_3|y_2)\tau$$

$$= \delta(y_3 - y_2) + W(y_3|y_2)\tau - W(y_2|y_3)\tau + O(\tau^2)$$

A Chemical reaction: $R_1 + R_2 \rightleftharpoons P_1 + P_2$  Fermi’s Rule: $\lambda_{if} = \frac{2\pi}{\hbar}|M_{if}|^2\rho_f$
The probability of a Markov process depends only on the probability of the last process

\[ \mathcal{P}(y_n|y_{n-1}, y_1) = \mathcal{P}(y_n|y_{n-1}) \]

We assume a stationary (\( \mathcal{P}(t) = \mathcal{P}(t + \tau) \)) and homogeneous process (\( \mathcal{P}(t_1, t_2) = \mathcal{P}(t_1 - t_2) \)) (a more general case is the Boltzmann equation) The conditional probability between the state \( y_3 \) and \( y_1 \) can be written:

\[ \mathcal{P}(y_3|y_1) = \int \mathcal{P}(y_3|y_2)\mathcal{P}(y_2|y_1)\,dy_2 \]

One step process: Brownian motion, shot noise, decay.
We derive the conditional probability with respect to the first order
\[ \partial_{t} P(y_3|y_2) = W(y_3|y_2) - W(y_2|y_3) \]
The derivative of the conditional probability is:
\[
\partial_{t} P(y_3|y_1) = \int \left( W(y_3|y_2)P(y_2|y_1) - W(y_2|y_3)P(y_3|y_1) \right) dy_2
\]
We remove the stationarity condition, multiply by \( P(y_1, t) \) and integrate over \( x_1 \)

\[
d_{t} P(y, t) = \int W(y, y')P(y', t) - W(y', y)P(y, t) dy'
\]

Einstein coefficients: Spontaneus emission \( A_{mn} \), Absorption: \( B_{nm}J \), Stimulated emission: \( B_{mn}J \)
Discrete master equation

A chemical reaction generation/ricombination (discrete, nonlinear) \( X \xrightleftharpoons{W \searrow W^\dagger} 2X \):

\[
\dot{P}_n = W^\dagger n(n+1)P_{n+1} + W(n-1)P_{n-1} - W^\dagger nP_n - Wn(n-1)P_n
\]

The master equation says that the probability of a transition in a time \( t \) is the sum of the gain in changing from \( m \rightarrow n \) minus the loss between \( n \rightarrow m \)

In the steady state the lhs of the master equation is zero \( \sum_m W_{nm}P_m = (\sum_m W_{mn})P_n \)

Detailed balance: \( W_{nm}P_m = W_{mn}P_n \)

Detailed balance is necessary but not sufficient for the equilibrium (microscopic reversibility)
**Fokker-Planck equation**

The evolution of a single event $\mathcal{P}(t)$ is governed by the master equation which describes all statistical properties.

We recast the integro equation of a master equation into the form of a Kramers-Moyal expansion ($r = x - x'$, small)

$$
\partial_t \mathcal{P}(x, t) = \mathcal{P}(x, t) \int W(x|r)\mathcal{P}(x, t) - W(x - r)\mathcal{P}(x, t)dr \\
- \int r\partial_x(W(x|r)\mathcal{P}(x, t))dr + \frac{1}{2} \int r^2\partial_x^2(W(x|r)\mathcal{P}(x, t))dr \pm \ldots
$$

or

$$
\dot{\mathcal{P}}_t(x) = \sum_n \frac{-n}{n!}\partial^n_x D_{KM}^{(n)} \mathcal{P}_t(x) \quad D_{KM}^{(n)}(x, t) = \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau} < (x(t + \tau) - x(t))^n >
$$

From the Pawula Theorem one can either use the first moment, the first and the second or every of them. In Most of the case two moments are enough, (e.g. Gaussian noise):

Fokker-Planck equation

$$
\dot{\mathcal{P}}(x, t) = -\partial_x D^{(1)} \mathcal{P}(x, t) + \partial_x^2 D^{(2)} \mathcal{P}(x, t)
$$
We consider the equation:

\[ \dot{x}(t) = X(x) + \xi(t) \]

The moments of the noise \( \xi \) are connected with the Kramers-Moyal expansion

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\tau}^{\tau + \Delta} ds < \xi(s) | x(t) = x_0 > = D_{KM}^{(1)} - X(x)
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\tau}^{\tau + \Delta} dt_1 \ldots dt_n < \xi(t_1) \ldots \xi(t_n) | x(t) = x_0 > = D_{KM}^{(n)}
\]

The first moment is deterministic, the higher order stochastic (mesoscopic description)

Considering the same moments the following SDEq is equivalent to the previous MEq

A non zero mean of the noise contributes to a drift term

From the Fokker-Planck equation we define a flux

\[ J := -D^{(1)} X(x) \mathcal{P}(x) + \frac{1}{2} D^{(2)} \mathcal{P}(x) \]

The FPEq can be written as a conservation law:

\[ \partial_t \mathcal{P}(x) - \partial_x J(x) = 0 \]

Typical boundary condition

- Natural (decay): \( \int \mathcal{P}(x) = \text{norm} \)
- Reflecting (wall): \( J(x = a, t) = 0 \)
- Absorbing (first passage time): \( P(x = a, t) = 0 \)
Neuron dynamics, role of noise

The dynamics of the voltage in a neuron is ruled by the equation:

\[ \dot{V}(t) = -\frac{V(t)}{\tau} + I(t) \quad I(t) = \mu + \sigma_w \eta(t) + \sigma_w \frac{\beta}{\sqrt{2\tau_c}} z(t) \]

Where \( \eta(t) \) is a white noise and \( z(t) \) is an auxiliary colored noise, \( V \) is a potential between \( H \) and \( \Theta \)

\[
C(t, t') = \langle (I(t) - \langle I(t) \rangle)(I(t') - \langle I(t') \rangle) \rangle = \sigma_w^2 \delta(t - t') + \frac{\sum_2}{2\tau_c} e^{-\frac{|t-t'|}{\tau_c}}
\]

The Fokker-Planck equation is:

\[
\left( \partial_V \left( f(V) - \mu + \frac{\sigma^2}{2} \partial_V \right) + \frac{1}{\tau_c} \partial_z (z + \partial_z) - \sqrt{\frac{2\sigma^2 \alpha^2}{\tau_c}} \partial_V \right) P = -\delta(V - H) J(z)
\]

The firing rate \( \nu \) is the probability per unit time that the potential cross a threshold \( \Theta \). \( J(z) \) is the escape probability current.
The firing rate $\mathcal{P}(V, z)$ is the steady state probability. We suppose $\tau_c < \tau_{\text{ref}}$ (correlation, refractory).

\[
J(z) = \frac{\nu_{\text{out}}}{\sqrt{2\pi}} e^{-x^2/2}
\]

$z$ after a spike relaxes to the stationary distribution. The Fokker-Planck has to be resolved with the normalisation condition

\[
\nu_{\text{out}} \tau_{\text{ref}} + \int_{-\infty}^{\Theta} dV \int_{-\infty}^{\infty} dz \mathcal{P}(V, z) = 1
\]

Finally the output firing rate is given by

\[
\nu_{\text{out}} = \int dz J(z)
\]
Summary

- Transition prob
- MEq
- Stochastic
- Langevin
- Continuous
- Noise distribution
- Kramers-Moyal
- Two moments
- Fokker-Planck
- Dissipative
- Non-gaussian noise
- Olstein-Uhlenbeck
- Neuronal model
Thank you for your attention

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References
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Langevin equation

\[ \dot{x} = X(x) + \xi(t) \]

The probability distribution \( P(y, t) \) is defined as:

\[ P(y, t) = \langle \delta(y - x(t)) \rangle_{\xi} \]

Advection (drift) term \( D^{(1)} \): \( \dot{x} = X(x) \)

\[
\partial_t P(y, t) = -\dot{x} d_y \delta(y - x) = -X(x) d_y \delta(y - x) = -d_y (\delta(y - x) F(x)) \\
= -d_y (\delta(y - x) F(y)) = -d_y (F(y) P(y, t))
\]

Diffusive term \( D^{(2)} \): \( \dot{x} = \xi(t) \)

\[
P(x, t) = \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{y^2}{2\Delta t}} \quad \partial_t P(x, t) = \frac{D^{(2)}}{2} \partial_x^2 P(x, t) \\
\partial_t P(x, t) = -D^{(1)} \partial_x P(x, t) + D^{(2)} \partial_x^2 P(x, t)
\]
Ornstein-Uhlenbeck process

We start from the dissipative Langevin equation

$$\Delta v(t) = -\gamma v(t) + \sigma \Delta \xi(t)$$

The KM coefficients are $D^{(1)} = -\gamma v$, $D^{(2)} = \frac{\sigma^2}{2}$

The stationary solution is given by $\dot{P}(x) = 0$

$$P(v) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{v^2 m}{2k_B T}}$$